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Classical self-force on an electron near a charged, rotating black hole

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Abstract. There have been several investigations on the influence of a gravitational field on charged test particles. This note demonstrates a very concise method to determine the force on a charged particle on the $\theta = 0$ axis of a black hole. The relevance of this result to the emission of charged particles by mini black holes is discussed.

The gravitational field has been known to influence the electrostatic interaction of a charged particle in a way that the particle experiences a finite self-force. There have been quite a few approaches to demonstrate this effect (DeWitt and DeWitt 1969, Berends and Gastmans 1978, MacGruder 1978, Vilenkin 1979) most of which employ a weak field approximation to obtain an expression for a repulsive force on the particle. Smith and Will (1980) have obtained an exact expression for this force in the case of a Schwarzschild metric by computing the stress–energy tensor. Their calculation involves transforming into isotropic and freely falling coordinates resulting in a cumbersome algebra. The purpose of this note is to present an elegant method which confirms earlier results and also lends itself to a straightforward generalisation to more complicated metrics of the Kerr–Newman background. These cases would definitely be very difficult, if not impossible, to work out using the techniques in Smith and Will (1980).

We start with the Schwarzschild metric

$$ds^2 = -(1-2M/r)\,dt^2 + (1-2M/r)^{-1}\,dr^2 + r^2\,d\Omega^2 \tag{1}$$

for which the potential due to a charge, e, at rest on the axis $\theta = 0$ at $r = r'$ is given by (Gibbons, unpublished)

$$A_0 = \frac{e}{rr'}\frac{(r-M)(r'-M) - M^2\cos\theta}{[(r-M)^2 + (r'-M)^2 - 2(r-M)(r'-M)\cos\theta - M^2\sin^2\theta]^{1/2}} + \frac{eM}{rr'}. \tag{2}$$

The first part of this expression was determined by Copson (1928). This solution has a source of strength $-eM/r'$ at $r = 0$ in addition to the source at $r = r'$. In order to construct a potential with just one source at r', Linet (1976) added the second term in equation (2) as the suitable monopole field to obtain an expression which is non-singular everywhere except at r'. The work done to bring up a charge de to a point r on the axis is therefore

$$dW = e\,de\left[\frac{1}{|r-r'|}\left(1 - \frac{M}{r'} - \frac{M}{r}\right) + \frac{M}{r'r}\right]. \tag{3}$$

In the limit $r \to r'$, the total energy required to assemble a charge e is

$$W = \int \mathrm{d}W = \frac{e^2}{2}\frac{\sqrt{-g_{00}}}{|\delta|} + \frac{1}{2}\frac{Me^2}{r'^2}\frac{\delta}{|\delta|} + \frac{1}{2}\frac{Me^2}{r'^2} \tag{4}$$

where $\delta = (1-2M/r)^{-1/2}(r-r')$. The first term is the red-shifted self-energy of a charge of size $|\delta|$ and can be absorbed into the bare mass of the particle. The second term, which depends on the sign of δ, averages out to zero for a spherically symmetric assembly of the charge. The third term, coming directly from the Linet image correction, leads to the inverse cube force we sought for. This force may therefore be interpreted as being due to the image charge induced inside the hole. However, there is one essential difference: the force in our case is repulsive and not attractive.

The same computation is easy to repeat for a charged particle in the Reissner–Nordström background. The solution for the potential is (Linet and Léauté 1976)

$$A_0 = \frac{e}{rr'}\frac{(r-M)(r'-M)-(M^2-Q^2)\cos\theta}{[(r-M)^2+(r'-M)^2-2(r-M)(r'-M)\cos\theta-(M^2-Q^2)\sin^2\theta]^{1/2}} + \frac{Me}{rr'}, \tag{5}$$

Q being the charge of the hole. The energy W required to assemble the charge e on the axis is thus found to be

$$W = m\sqrt{-g_{00}} + eQ/r' + \tfrac{1}{2}Me^2/r'^2. \tag{6}$$

To find the static solutions for Maxwell's equations in the Kerr–Newman background, one starts by adopting the Kinnersley (1969) tetrad for the metric, for which the tetrad vectors have the following (t, r, θ, φ) components:

$$\begin{aligned} l_\mu &= (-1, \rho'^2/\Delta, 0, a\sin^2\theta), \qquad \eta_\mu = (1/2\rho'^2)(-\Delta, \rho'^2, 0, \Delta a \sin^2\theta), \\ m_\mu &= [2^{1/2}(r + \mathrm{i}a\cos\theta)]^{-1}(-\mathrm{i}a\sin\theta, 0, \rho'^2, \mathrm{i}\sin\theta(r^2+a^2)), \end{aligned} \tag{7}$$

where $\Delta = r^2 - 2Mr + a^2 + Q^2$, a and Q being the angular momentum and charge parameters of the hole, and $\rho'^2 \equiv r^2 + a^2\cos^2\theta$. The non-vanishing spin coefficients are

$$\begin{aligned} &\rho = -(r - \mathrm{i}a\cos\theta)^{-1}, \qquad \beta = -\frac{\bar\rho\cot\theta}{2\sqrt{2}}, \qquad \pi = \frac{\mathrm{i}a\rho^2\sin\theta}{\sqrt{2}}, \qquad \alpha = \pi - \bar\beta, \\ &\tau = -\frac{\mathrm{i}a\rho\bar\rho\sin\theta}{\sqrt{2}}, \qquad \mu = \frac{\rho^2\bar\rho\Delta}{2}, \qquad \gamma = \mu + \rho\bar\rho\frac{(r-M)}{2}. \end{aligned} \tag{8}$$

This reduces the Newman–Penrose form for Maxwell's equations

$$\begin{aligned} &F^{\mu\nu}{}_{;\nu} = 4\pi J^\mu, \qquad \phi_0 \equiv F_{\mu\nu}l^\mu m^\nu, \\ &\phi_1 \equiv \tfrac{1}{2}F_{\mu\nu}(l^\mu\eta^\nu + \bar m^\mu m^\nu), \qquad \phi_2 \equiv F_{\mu\nu}\bar m^\mu n^\nu, \end{aligned} \tag{9}$$

to

$$\frac{\partial\Phi_1}{\partial r} + \frac{1}{\sqrt{2}\chi}\frac{\partial\Phi_0}{\partial\theta} = 2\pi\rho^{-2}J_l, \tag{10a}$$

$$\frac{\partial\Phi_1}{\partial r} - \frac{\sqrt{2}\rho'^2\bar\rho}{\Delta\rho\sin\theta}\frac{\partial\Phi_2}{\partial\theta} = \frac{4\pi\rho^{-2}\rho'^2}{\Delta}J_n, \tag{10b}$$

$$\frac{\partial\Phi_1}{\partial\theta} - \frac{\Delta}{\sqrt{2}\chi}\frac{\partial\Phi_0}{\partial r} = -\frac{2\sqrt{2}\pi\rho^{-2}}{\bar\rho}J_m, \tag{10c}$$

$$\frac{\partial \Phi_1}{\partial \theta}+\frac{\sqrt{2}}{\rho^2 \sin\theta}\frac{\partial \Phi_2}{\partial r}=2\sqrt{2}\pi\rho^{-3}J_{\bar{m}}, \tag{10d}$$

where

$$\chi \equiv \Delta\rho^2 \sin\theta, \qquad \Phi_0 \equiv \rho^{-1}\chi\phi_0, \qquad \Phi_1 \equiv \rho^{-2}\phi_1, \qquad \Phi_2 \equiv \rho^{-1}\sin\theta\,\phi_2.$$

From (10) we obtain $\Phi_0 = -2\Phi_2 + \text{constant}$, assuming regularity of the field on the axis,

$$\phi_0 = -2\phi_2/\Delta\rho^2; \tag{11}$$

thus ϕ_0 and ϕ_1 suffice to specify the field.

Following Teukolsky (1973) and Misra (1977), we may define $\psi = \phi_2/\rho^2$ to give

$$\Delta\frac{\partial^2\psi}{\partial r^2}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)-\frac{\psi}{\sin^2\theta}=-4\pi J \tag{12}$$

with

$$J=\left(-\frac{\Delta}{2}\frac{\partial}{\partial r}+\frac{\rho\Delta}{2}\right)(\rho^{-2}J_{\bar{m}})-\left(-\frac{\Sigma\rho}{\sqrt{2}}\frac{\partial}{\partial\theta}+\frac{\mathrm{i}a\sin\theta}{\sqrt{2}}(\rho'\rho^2-1)\right)\rho^{-2}J_n. \tag{13}$$

We can now obtain the solution to (12) for a charge e on the axis for which $J^t=(e/2\pi\rho'^2)\delta(r-r')\delta(\cos\theta-1)$. To do this we define $\psi=\partial\xi/\partial\theta$ and introduce a new radial coordinate $r\equiv R+M+[M^2-(a^2+Q^2)]/4R$ to give

$$R^2\frac{\partial^2\xi}{\partial R^2}-\frac{(M^2-a^2-Q^2)/2R}{1-(M^2-a^2-Q^2)/4R^2}\frac{\partial\xi}{\partial R}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\xi}{\partial\theta}\right)=0. \tag{14}$$

The solution to this equation is obtained by direct integration after introducing a new variable

$$x=[R^2+R^2(r')-2R(r')R\cos\theta][R^2-\tfrac{1}{4}(M^2-a^2-Q^2)]^{-1}.$$

The result is

$$\xi=\frac{c_1[(r-M)(r'-M)-(M^2-a^2-Q^2)\cos\theta]}{[(r-M)^2+(r'-M)^2-2(r-M)(r'-M)\cos\theta-(M^2-a^2-Q^2)\sin^2\theta]^{1/2}}+c_2. \tag{15}$$

The constants c_1 and c_2 can be obtained by integrating the solution across the source. From equation (12), we obtain $c_1=-(e/2\sqrt{2})(r'+\mathrm{i}a)^{-1}$, c_2 being arbitrary. Thus the solution is

$$\begin{aligned}\xi&=-\frac{e}{2\sqrt{2}(r'+\mathrm{i}a)}\\&\quad\times\frac{(r-M)(r'-M)-(M^2-a^2-Q^2)\cos\theta}{[(r-M)^2+(r'-M)^2-2(r-M)(r'-M)\cos\theta-(M^2-a^2-Q^2)\sin^2\theta]^{1/2}}+c_2\\&\equiv-[e/2\sqrt{2}(r'+\mathrm{i}a)]x+c_2.\end{aligned} \tag{16}$$

This gives

$$\phi_2=-[e\rho^2/2\sqrt{2}(r'+\mathrm{i}a)]\partial x/\partial\theta.$$

The Maxwell equations (10*b*) and (10*d*) give

$$\frac{\partial}{\partial\theta}\left(\frac{\phi_1}{\rho^2}\right)=\frac{e}{2(r'+\mathrm{i}a)}\left[\frac{\partial}{\partial\theta}\left(\rho^{-1}\frac{\partial x}{\partial r}\right)+\mathrm{i}a\sin\theta\frac{\partial x}{\partial r}+\frac{\partial x}{\partial\theta}\right]. \tag{17}$$

Integrating with respect to θ gives

$$\frac{\phi_1}{\rho^2} = \frac{e}{2(r'+\mathrm{i}a)}\left(\rho^{-1}\frac{\partial x}{\partial r} + x + \mathrm{i}a\frac{\partial y}{\partial r}\right) + f(r) \tag{18}$$

where

$$Y^2 = (r-M)^2 + (r'-M)^2 - 2(r-M)(r'-M)\cos\theta - (M^2-a^2-Q^2)\sin^2\theta.$$

The function $f(r)$ is found by examining the large-distance behaviour of the field—or equivalently by appealing to Gauss's theorem. To this effect we invert equation (9), i.e.

$$F_{\mu\nu} = 2[\phi_1(n_{[\mu}l_{\nu]} + m_{[\mu}\bar{m}_{\nu]}) + \phi_2 l_{[\mu}m_{\nu]} + \phi_0 \bar{m}_{[\mu}n_{\nu]} + \mathrm{CC} \tag{19}$$

to give $F_{tr} = -2\,\mathrm{Re}\,\phi_1$ at a point on the Z axis, i.e. $\theta = 0$. Requiring the far-away field to correspond to a charge at $r = r'$ and the black hole charge Q inside the horizon gives, from equation (18),

$$f(r) = eM/2(r'+\mathrm{i}a) + \tfrac{1}{2}Q. \tag{20}$$

The force on the charge e would be given by $eF_{tr} = \lim_{r\to r'}(-2e\,\mathrm{Re}\,\phi_1)$. As before, we average out terms depending on the sign of $(r-r')$ to zero, giving

$$\text{Force} = \frac{e^2}{|r-r'|}\left(-\frac{(r'-M)}{r'^2+a^2} + \frac{r'[(r'-M)^2-(M^2-a^2-Q^2)]}{(r'^2+a^2)^2}\right) + \frac{r'}{(r'^2+a^2)^2}e^2M + \frac{Qe(r'^2-a^2)}{(r'^2+a^2)^2}. \tag{21}$$

$e^2/|r-r'|$ can again be absorbed in the bare mass of the particle. The second term is the repulsive force we were looking for and the third term is just the electrostatic force between the charges Q of the black hole and e of the particle.

The expression for the force in equation (21) modifies the effective potential used to analyse the black hole emission process by terms proportional to e^2. This would not substantially affect the emission of charged particles from highly charged holes, but it might affect the emission from holes with a small charge $Q = Ze$ (say). For a non-rotating uncharged hole, the repulsion can overcome gravity at distances $r \approx e^2/m$, i.e. the classical electron radius. This is outside the horizon for small black holes ($M < 10^{16}$ g). The chemical potential of the hole is raised (for both signs of the charge) by a value e^2M/r_+^2. All the resulting corrections could be included in the calculation of the thermal flux by computing the emission rates using the appropriate wave equations coupled to an effective electromagnetic field

$$A_0^{\mathrm{eff}} = Q/r + e(2Mr - Q^2)/2r^3. \tag{22}$$

A_0^{eff} is not a solution to Maxwell's equations, but it is divergence free. The absorption coefficients could now be computed numerically following Page (1977). For a Schwarz-schild hole, the chemical potential is $e^2/8Mm$ times the rest mass of the particle. Thus the emission of charged particles would be suppressed relative to the emission of neutral particles by this effect. If $Z = 1$, the Coulomb potential is between four and two times smaller than the Linet potential, depending on how highly charged the hole is. If the hole is neutral the ratios of rest mass to thermal energy and Linet potential energy to thermal energy are $8\pi Mm$ and πe^2 respectively. The energy required to overcome the Linet potential is always small compared with thermal energies. This indicates that the suppression is never very effective. In any case there are other effects which enter at this order in the fine structure constant (Page 1977).

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